BRAINIACS
BRAIN IMAGING AND COMPUTING SCIENCES

2025 Volume 6
Issue 1 Edoc T43E35E87

Topological Neural Coding*

Andrés Pomi†

Abstract

Understanding the neural representation of complex cognitive activities, such as processing algebraic or topological structures like graphs, groups, and knots, is a fundamental challenge in cognitive neuroscience. This study explores how associative matrix memories, as mesoscopic models, bridge symbolic data processing with dynamic neuronal activity. We demonstrate that these memories naturally represent graphs of associations between concepts and extend this framework to encode finite groups via their Cayley graphs and knots through tensor product representations. For knots, we propose a context-dependent associative memory matrix that captures crossing states in knot diagrams, linking Gauss codes to Seifert circles and aiding knot classification. These representations provide a unified neural framework for encoding diverse topological objects, offering insights into the brain's ability to process abstract mathematical structures.

Keyphrases

Associative memories, representation models, tensor products, graphs, groups, knots.

Introduction

A theory of cognition must account for the neuronal representation of all cognitive activity. When our minds consider algebraic or topological structures, such as graphs, groups, or knots, how are they represented in our brain from a neurocognitive perspective? What is the neural encoding of these cognitive activities?

According to the current paradigm of cognitive neuroscience, perception, thought, memory or any cognitive function correspond to configurations of activity of large neuronal groups that integrate distributed circuits of interconnected brain areas. Associative matrix memories are classical mathematical models of cognitive brain activity that fit perfectly into this scenario of brain functionality (Kohonen 1977). They are mesoscopic models that link the level of algorithms operating with complex symbolic data with the underlying dynamic neuronal level. In these models, symbolic expressions and operations are represented by states and transformations in abstract vector spaces (Graben and Potthast 2009). The activity patterns of large neuronal groups distributed throughout the brain are mapped onto vectors. The basic activity of any group of neurons is modeled as associations between the configuration of activity reaching the neuronal group (an input vector $\mathbf{f}$) and the pattern of neuronal activity with which this group of neurons responds to its input (an output vector $\mathbf{g}$). These associations between vectors representing cognitive states are realized by matrix memories whose coefficients are real numbers attributed to synaptic strength (Anderson 1995). Assuming quasi-orthogonality for distinguishable patterns and Hebbian synapses, these matrix memories assume the simple outer product rule:

$$\mathbf{M} = \sum_{i=1}^{N} \mathbf{g}_i \mathbf{f}_i^{\top} \tag{1}$$

Context-dependent associative memories, endowed with tensor product representation for input compositionality allows for adaptive associations (Mizraji 1989; Pomi-Brea and Mizraji 1999), becoming a powerful neural tool to overcome the divorce between symbolic models and neural networks.

$$\mathbf{E} = \sum_{i} \sum_{j} \mathbf{g}_{ij} (\mathbf{p}_{ij} \otimes \mathbf{f}_i)^{\top} \tag{2}$$

The great advantage of this approach is that it preserves the linear algebra representation, which allows mathematical operations to advance an algebraic theory of cognition (Mizraji 2008; Graben and Potthast 2009).

In this communication, we present a research program that seeks to find neural representations of the abstract, algebraic, and topological structures of mathematics within the framework of context-dependent associative memories, and we show how the tensor product acts as a structuring element of the space. First, we show how matrix associative memories naturally support a graph representation of their stored semantic structure. Then, we present a possible neural representation of mathematical group structures based on associative memory models that store finite groups through their Cayley graphs. After reviewing these results, we introduce here a neural representation of knots within the framework of this theory.

Graphs of Associations

Human memory is inherently associative, with its semantic content represented by psychologists as graphs since the 1960s (Spitzer 1999). The question arises: how does the brain support these structures? We have demonstrated that associative matrix memories naturally represent graphs of associations between concepts (Pomi and Mizraji 2004).

*Report presented 2025-06-02, Neural Coding 2025, 16th International Neural Coding Workshop, Ascona Switzerland.

†Correspondence to pomi@fcien.edu.uy.

When concepts are encoded in the neural domain by approximately orthogonal vectors, the adjacency matrix of the association graph, $\mathbf{A}(\Gamma)$, is the same neural memory (denoted $\mathbf{M}_w$) when encoded in the standard basis ($\mathbf{M}_e$). Hence, these two matrices are similar, sharing the same spectrum of eigenvalues: $\{\lambda(\mathbf{M}_w)\} = \{\lambda(\mathbf{M}_e)\}$, where $\mathbf{M}_e$ is a block matrix with $\mathbf{A}(\Gamma)$ in the upper left diagonal block and zeros elsewhere. In the reduced space with the dimension of the number of concepts in the memory, and k the number of associations, $\mathbf{M}_e = \sum_{i=1}^{k} \mathbf{e}_i \mathbf{e}_i j^\top = \mathbf{A}(\Gamma)^\top$. The associative graph and its spectrum are code-invariant properties of the memory. For context-sensitive associations, a multidigraph is formed, where specific contexts select subgraphs of associations active under that context.

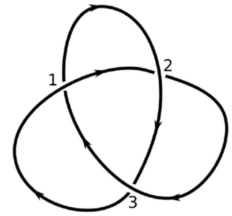
These findings imply that, without knowing the precise neural vector coding or its dimension for each concept in an individual brain, exploring the associations contained in memory can reveal the structure of its association graph. Since the adjacency matrix of the graph is similar to the neural memory matrix, we gain access to the spectrum of eigenvalues of the real, yet unknown, neural memory! This opens the possibility of modifying cognitive dynamics through targeted modifications to the graph, enabling a form of cognitive engineering or scientific psychotherapy.

Associative Memories Encoding Groups

Groups, as mathematical structures, underpin diverse aspects of nature, including notions of beauty and symmetry. Beyond their multiplication tables, finite groups can be uniquely characterized through a *presentation* and its associated Cayley graph. Let G be a group and S a set of its elements. If 'all' elements of G can be expressed as products of elements in S and their inverses, the elements of S are termed *generators* of G. For a group G with generating set S, the Cayley color graph is constructed as follows: each element $g_i \in G$ is assigned a vertex v_i, and each generator $s_i \in S$ is assigned a color c_i. Then, there is a directed edge of color c_i connecting v_1 to v_2 if $g_2 = g_1 \cdot s_i$. Closed walks within the graph correspond to *relations* defined on the generating set, where a *relation* is a word that evaluates to the identity element in G. The characterization of a finite group by its generators and a minimal set of relations, sufficient to imply all relations in G, is called a *presentation* of the group. We have proposed a novel context-dependent associative matrix for representing groups, capturing their structure in a neural framework (Pomi 2016): $\mathbf{E} = \sum_k \sum_i \mathbf{g}'_{i(k)} (\mathbf{g}_i \otimes \mathbf{s}_k)^\top$. By coding the elements of the group with vectors of the standard basis, we obtain the expression $\mathbf{E} = \sum_{s \in S} \mathbf{A}(\Gamma_s) \otimes \mathbf{s}^\top$, where the action of the context vectors $\mathbf{s}$ (generators) is to dissect the monochromatic subgraphs Γ_s of the Cayley graph. The adjacency matrices of these monochromatic subgraphs store the transitions between elements of the group under the action of each generator. These matrices that represent generators are permutation matrices with the same dimension of the order of the group. Group theorists know this faithful matrix representation of a group as its *regular representation*. These adjacency matrices generate the other elements by fulfilling the defining relations of the presentation of the group.

A Neural Representation of Knots

Knots are immersions of simple closed curves in $\mathbb{R}^3$, and their graphic representation is usually achieved by projection onto the plane. The mathematical theory of knots is framed within topology and has undergone intense and interesting development over the last fifty years (Adams 1994). Links have been established between knot theory and groups, graphs and braids, among other areas, as well as deep relationships with several fields of fundamental physics (Kauffman 2001).



$$M(3_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure 1: Diagram of the Trefoil knot and its associative matrix $\mathbf{M}(3_1)$ coded in the canonical basis according to equation 4. Note that the 1s in the matrix are disposed in the antidiagonal blocks, corresponding to the adjacency matrix of a bipartite graph between the set of three Over vertices and the set of three Under ones.

In this communication, we represent a knot as a context-dependent associative memory matrix, a mathematical object originating within the theory of information processing in neural systems. We use a tensor-product representation for each *position* in a knot diagram, defined by a *state* (Over or Under) at each crossing, with its N crossings and the two states represented by orthonormal basis vectors. Given a crossing x_i and a state s_j, a position is encoded by the tensor product $\mathbf{s}_j \otimes x_i$, where $\mathbf{x}_i$ are the vectors embedding the crossings and $\mathbf{s}_j$ are those embedding the states. The *associative matrix* $\mathbf{M}(K)$ of a knot diagram (Pomi 2013) stores transitions between the $2N$ successive positions in a given orientation of the knot and is defined as

$$\mathbf{M}(K) = \sum_{i=1}^{N} \sum_{j=1}^{2} (\mathbf{s}'_{ij} \otimes \mathbf{x}'_{ij})(\mathbf{s}_j \otimes \mathbf{x}_i)^\top \tag{3}$$

where $\mathbf{s}_j, \mathbf{s}'_{ij} \in \mathbb{R}^2$ and $\mathbf{x}_i, \mathbf{x}'_{ij} \in \mathbb{R}^N$ are all vectors belonging to their respective orthonormal bases, and $\mathbf{M}(K)$ is a square matrix of dimensions $2N \times 2N$. When the orthonormal vectors belong to the canonical basis, $e_1 = [1, 0, \ldots, 0]^\top, e_2 = [0, 1, \ldots, 0]^\top, \ldots$ represent crossings x_i, and $\mathbf{O} = [1, 0]^\top$, $\mathbf{U} = [0, 1]^\top$ for states s_i (Over and Under), the associative matrix becomes a permutation matrix.

As an illustration, let us write the associative matrix for a minimal oriented diagram of the Trefoil knot (3_1):

$$\begin{aligned} \mathbf{M}(3_1) = {} & (\mathbf{U} \otimes \mathbf{e}_2)(\mathbf{O} \otimes \mathbf{e}_1)^\top + (\mathbf{O} \otimes \mathbf{e}_3)(\mathbf{U} \otimes \mathbf{e}_2)^\top \\ & + (\mathbf{U} \otimes \mathbf{e}_1)(\mathbf{O} \otimes \mathbf{e}_3)^\top + (\mathbf{O} \otimes \mathbf{e}_2)(\mathbf{U} \otimes \mathbf{e}_1)^\top \\ & + (\mathbf{U} \otimes \mathbf{e}_3)(\mathbf{O} \otimes \mathbf{e}_2)^\top + (\mathbf{O} \otimes \mathbf{e}_1)(\mathbf{U} \otimes \mathbf{e}_3)^\top \end{aligned} \tag{4}$$

See the diagram of the Trefoil and the matrix coded in the canonical basis in Figure 1.

When presented with an arbitrary position ($\mathbf{s}_j \otimes \mathbf{x}_i$), the matrix memory retrieves the tensor composition representing the next position (crossing and state), given the orientation of the knot diagram. Reinjecting this output as a new entry, the associative matrix of the knot yields, in a successive manner, all the positions of the oriented diagram until the completion of the tour. Hence, the matrix $\mathbf{M}$ stores all the information needed to rebuild the knot. Renaming the tensor compositions according to the state and crossing number, for example $O \otimes e_1 \equiv O1$, we obtain $O1\, U2\, O3\, U1\, O2\, U3\, O1 \ldots$, which is the

Gauss code of the Trefoil knot, with $O1, U2, \ldots$ being referred to as *letters*.

This matrix $\mathbf{M}(K)$, a permutation matrix with twice the dimension of the crossing number, is structured in four square blocks, two "diagonal" matrices of dimension $N \times N$, corresponding to $O - O$ and $U - U$ transitions, and two "antidiagonal" corresponding to alternating transitions $U - O$ and $O - U$:

$$\mathbf{M}(K) = \mathbf{UO}^\top \otimes \sum_i \mathbf{x}_{i+1}\mathbf{x}_i^\top + \mathbf{OU}^\top \otimes \sum_j \mathbf{x}_{j+1}\mathbf{x}_j^\top + \mathbf{UU}^\top \otimes \sum_h \mathbf{x}_{h+1}\mathbf{x}_h^\top + \mathbf{OO}^\top \otimes \sum_k \mathbf{x}_{k+1}\mathbf{x}_k^\top \quad (5)$$

When the knot is of alternating type, it corresponds to the adjacency matrix of a bipartite graph between the set of N Over vertices and the set of N Under ones (see Figure 1).

From any diagram, a set of circles, called Seifert circles, can be obtained by eliminating the crossings (Murasugi 1996). For alternating knots, the anti-diagonal submatrices also represent the Seifert circles of the knot; for non-alternating diagrams, Seifert circles partially coincide with the matrices (Bosch 2019). Thus, this associative matrix of a knot creates a link between its Gauss code and the Seifert circles, and provides certain tools for knot classification. In particular, an advanced representation, making use of the adjacency matrix of the "Gauss diagram" (Polyak and Viro 1994), can distinguish between the two Trefoil knots, as was demonstrated by Bosch (2019).

Conclusion

Context-sensitive associative memories are capable of displaying elegant and interrelated neural representations of different topological objects, such as graphs, finite groups, and knots. These results are promising for the consolidation of this type of vector model of neural states as a universal algebraic representation of cognition, extendable to other products of human culture.

Citation

Brainiacs 2025 Volume 6 Issue 1 Edoc T43E35E87
Title: "Topological Neural Coding"
Authors: Andrés Pomi
Dates: created 2025-05-01, presented 2025-06-02, published 2025-07-16,

Contact: pomi@fcien.edu.uy
URL: BrainiacsJournal.org/arc/pub/Pomi2025TopoNC
PDP: /Nexus/Brainiacs/Pomi2025TopoNC
DOI: /10.48085/T43E35E87

Affiliations

Andrés Pomi, Sección Biofísica y Biología de Sistemas, Facultad de Ciencias, Universidad de la República, Montevideo, Uruguay.

References

[1] C. C. Adams. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots*. New York: W. H. Freeman, 1994. ISBN: 978-0-7167-2393-6 (cited p. 2).

[2] J. A. Anderson. *An Introduction to Neural Networks*. Cambridge: MIT Press, 1995 (cited p. 1).

[3] S. A. Bosch. "La teoría de nudos en biología: Exploración de una nueva representación matricial inspirada en un modelo de memorias asociativas." B.S. Thesis. Uruguay: Universidad de la República, 2019 (cited p. 3).

[4] P. beim Graben and R. Potthast. "Inverse problems in dynamic cognitive modeling." *Chaos: An Interdisciplinary Journal of Nonlinear Science* 19 (2009), p. 015103. DOI: 10.1063/1.3075222 (cited p. 1).

[5] L. H. Kauffman. *Knots and Physics*. 3rd ed. Singapore: World Scientific, 2001. ISBN: 978-981-02-4111-7 (cited p. 2).

[6] T. Kohonen. *Associative Memory: A System-Theoretical Approach*. Berlin: Springer, 1977. ISBN: 978-3-540-08266-8 (cited p. 1).

[7] E. Mizraji. "Context-dependent associations in linear distributed memories." *Bulletin of Mathematical Biology* 51 (1989), pp. 195–205 (cited p. 1).

[8] E. Mizraji. "Neural memories and search engines." *International Journal of General Systems* 37 (2008), pp. 715–738. DOI: 10.1080/03081070802173691 (cited p. 1).

[9] K. Murasugi. *Knot Theory and Its Applications*. Boston: Birkhäuser, 1996 (cited p. 3).

[10] M. Polyak and O. Viro. "Gauss diagram formulas for Vassiliev invariants." *International Mathematics Research Notices* 1994 (1994), pp. 445–453 (cited p. 3).

[11] A. Pomi. "Una representación matricial de nudos construida a partir de un modelo de memoria asociativa." In: *2as. Jornadas de +Biofísica*. Montevideo, 2013 (cited p. 2).

[12] A. Pomi. "A possible neural representation of mathematical group structures." *Bulletin of Mathematical Biology* 78 (2016), pp. 1847–1865. DOI: 10.1007/s11538-016-0184-5 (cited p. 2).

[13] A. Pomi and E. Mizraji. "Semantic graphs and associative memories." *Physical Review E* 70 (2004), p. 066136. DOI: 10.1103/PhysRevE.70.066136 (cited p. 1).

[14] A. Pomi-Brea and E. Mizraji. "Memories in context." *BioSystems* 50 (1999), pp. 173–188. DOI: 10.1016/s0303-2647(99)00005-2 (cited p. 1).

[15] M. Spitzer. *The Mind Within the Net: Models of Learning, Thinking, and Acting*. Cambridge, MA: MIT Press, 1999. ISBN: 978-0-262-19423-5 (cited p. 1).