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A Noisy Multi-Agent Framework for the Dynamics of Interacting Neurons*

Giuseppe D'Onofrio†

Abstract

This work investigates the collective behavior of neuronal populations through a stochastic multi-agent framework that models neurons as particles subject to noise in both the membrane voltage evolution and the interactions between neurons. Employing tools from stochastic calculus and measure theory, the study derives macroscopic equations that describe the mean-field dynamics of the system. These approximations enable a rigorous characterization of large-scale neural activity, including conditions for synchronization and stability in the presence of noise. The mathematical formulation is highly general, can be grounded in biologically plausible assumptions to offer insights into how local fluctuations can influence global patterns of brain dynamics. Potential applications include modeling cortical connectivity and analyzing neural variability observed in neuroimaging data. This approach establishes a formal link between microscopic neuronal interactions and emergent macroscopic behavior, providing a valuable analytical tool for theoretical neuroscience and the study of brain function under uncertainty.

Keyphrases

Mean-field limit; well-posedness of stochastic differential equations; neuronal modeling; random synaptic weights; integrate-and-fire.

Introduction

Starting from the seminal paper by Lapicque (Abbott 1999; Brunel and Rossum 2007), models of single neuron cells, seen as RC circuits, have spread and evolved over the century. Conductance-based models are biophysical representations of nerve cells, and the equations describing these models capture the generation of action potentials by focusing on the temporal dynamics of ion channels across the cell membrane. Prominent examples include the Hodgkin–Huxley model and the FitzHugh–Nagumo model and Morris–Lecar model. These models are nonlinear and biologically accurate, but their exact mathematical analysis is highly complex (Greenwood and Ward 2016). In parallel with the development of biophysical models, the introduction in linear models of sources of stochasticity, such as Poissonian inputs, has allowed researchers to study the overall behavior of the membrane potential without having to observe the dynamics of individual ion channels (Stein 1965; Tuckwell 1989). These models, often of the leaky integrate-and-fire type, were later made more mathematically tractable through diffusive approximations of the equations describing the temporal evolution of the membrane potential (Buonocore et al. 2010; Lánský 1984; Sacerdote and Giraudo 2013). At the same time, the aim of adding increasingly realistic biological features led to the development of more specific models, incorporating aspects such as the refractory period, inhibitory and excitatory reversal potentials (Lánská et al. 1994), and various types of synaptic connections (with weights also modeled as random variables in models with random synaptic weights (Grazieschi et al. 2019)). Moreover, to include the geometry of the neuron, which is usually assumed point-like, and to overcome certain limitations of diffusive approximations, jump-diffusion models have been proposed (Sirovich et al. 2013). Some of the most challenging issues involve incorporating adaptation and synchronization phenomena, typically associated with population models, into single neuron equations (Kobayashi et al. 2009). To regain mathematical tractability within these sophisticated models and in general nonlinear models, techniques such as averaging are often necessary. However, this comes with the risk of overlooking second-order phenomena (Ascione and D'Onofrio 2023).

To describe collective phenomena, systems of stochastic differential equations have been considered, in which the equations represent either sub-populations or individual units, which can make single-neuron models appear increasingly obsolete. Mean-field models take the approach of considering systems of equations for entire populations, but by reducing the dynamics to those of the mean field in the limit of a large number of agents, they ultimately allow for the study of a prototypical neuron's equation; effectively returning to the analysis of single-neuron dynamics (La Camera 2021). Traditional approaches to large neuronal networks often rely on diffusion approximations, which assume high-frequency, low-amplitude inputs. In contrast, mean-field theory provides a robust framework to analyze such systems without these assumptions, effectively reducing complex dynamics to a few key macroscopic parameters, such as firing rates.

In this framework, in (D'Onofrio and Melchor Hernandez 2025) we adapt multi-agent models that are typically used for simulating the actions and interactions of autonomous agents in crowds or leader–follower dynamics, to the context of neural modeling. We study a multi-population system in which the dynamics of each agent are governed by a system of stochastic differential equations within a general framework, driven by the collective state of the system. Each agent is associated with a probability measure that serves as a dynamic label, indicating the population to which it belongs. We make no assumption of prior

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†Correspondence to giuseppe.donofrio@polito.it.

knowledge about these labels, and we allow them to evolve as a result of interactions among agents. The system is further influenced by noise, affecting both the agents' positions and their labels.

Model

We first establish the well-posedness of the system and then investigate its mean-field limit as the number of agents tends to infinity and we analyze the properties of the resulting limit distribution.

Such models are traditionally applied to leader-follower dynamics, consensus formation, and control theory. Here, we propose a novel interpretation: modeling a network of interacting neurons whose synaptic weights, that describe the influence that a neuron has onto the others, follow a certain distribution in the spirit of neuronal networks with random synaptic weights.

The model falls within the class of *noisy integrate-and-fire* systems and is given by:

$$\begin{cases} X^i(t) = X_0^i + \int_0^t v_{\Lambda_s^N}(X^i(s), \lambda^i(s))ds + \sqrt{2\sigma}B_t^i, & \\ & \text{if } X^i(t^-) < X_F, \\ X^i(t) = X_R, & \text{if } X^i(t^-) = X_F, \\ \lambda^i(t) = \lambda_0^i + \int_0^t \mathcal{T}_{\Lambda_s^N}(X^i(s), \lambda^i(s))ds & \end{cases} \tag{1}$$

where the stochastic process X^i describes the time evolution of the membrane potential of a neuron receiving the inputs from the other neurons in the network it is embedded in. These contributions are summarized in the drift part of (1) according to the velocity field $v_{\Lambda_s^N}$ until the membrane potential reaches a physiological value X_F triggering a spike. As a result of this spike, which is assumed to occur instantaneously, the membrane potential of all the other connected neurons receives a contribution. Afterward, the dynamics of the spiking neuron restart from the resting state X_R. The strength of the interaction between neurons and its evolution are described through a general operator $\mathcal{T}_{\Lambda_t^N}$ that is perturbed by a random effect of the environment, being it dependent on X^i and has the following form:

$$\mathcal{T}(X^i(t), \lambda^i(t)) = \frac{1}{N}\sum_{k\geq 1}\sum_{i=1}^{N} \beta_k \alpha(u_k, X^i(t), \lambda^i(t)),$$

where β and α can be deterministic or stochastic according to the modeling purposes. This general formulation encompasses classical models with random synaptic weights as special cases. All quantities in (1) depend on Λ_t^N that is the empirical measure $\frac{1}{N}\sum_{i=1}^{N}\delta_{(X^i(t),\lambda^i(t))}$.

Discussion

This framework offers a flexible and mathematically rigorous approach to modeling complex interacting systems with evolving internal states, while preserving the tractability of single-neuron models. It opens avenues for studying emergent behavior in heterogeneous, noise-driven populations. The dependence of v on the state X^i follows the principles of integrate-and-fire neuronal models with reversal potentials. The dependence of v on λ^i, on the other hand, accounts for the different synaptic connections among neurons and could also include both inhibitory and excitatory contributions. Finally, the dependence of the whole dynamics on the empirical measure Λ, which in this case represents the distribution of membrane potential values, can be used to model phenomena such as collective behavior, bursting, and avalanches, to name a few. An open problem remains the analysis of the system for a large but finite number of agents, in order to determine whether the properties and terms preserved in the limiting case are indeed those that are biologically relevant (Helias et al. 2014).

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Contact: giuseppe.donofrio@polito.it
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Affiliations

Giuseppe D'Onofrio, giuseppe.donofrio@polito.it; Dipartimento di Scienze Matematiche, Politecnico di Torino, Italy.

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